

Systems of reaction-diffusion equations with spatially distributed hysteresis

Pavel Gurevich, Sergey Tikhomirov

1. Setting of the problem. Reaction-diffusion equations with spatially distributed hysteresis were first introduced in [1] to describe the growth of a colony of bacteria (*Salmonella typhimurium*) and explain emerging spatial patterns of the bacteria density. In [1, 2], numerical analysis of the problem has been carried out, however without rigorous justification. First analytical results were obtained in [3, 4] (see also [5]), where existence of solutions for multi-valued hysteresis was proved. Formal asymptotic expansions of solutions were recently obtained for some special case in [6]. Questions about the uniqueness of solutions and their continuous dependence on initial data as well as a thorough analysis of pattern formation remained open. In this paper, we formulate sufficient conditions that guarantee existence, uniqueness, and continuous dependence of solutions on initial data for systems of reaction-diffusion equations with discontinuous spatially distributed hysteresis. Analogous conditions for scalar equations have been considered by the authors in [7, 8].

Denote $Q_T = (0, 1) \times (0, T)$, where $T > 0$. Let $\mathcal{U} \subset \mathbb{R}^k$ and $\mathcal{V} \subset \mathbb{R}^l$ ($k, l \in \mathbb{N}$) be closed sets. We assume throughout that $(x, t) \in \overline{Q_T}$, $u(x, t) \in \mathcal{U}$, $v(x, t) \in \mathcal{V}$.

We consider the system of reaction-diffusion equations

$$\begin{cases} u_t = D\Delta u + f(u, v, W(\xi_0, u)), \\ v_t = g(u, v, W(\xi_0, u)) \end{cases} \quad (1)$$

with the initial and boundary conditions

$$u|_{t=0} = \varphi(x), \quad v|_{t=0} = \psi(x), \quad (2)$$

$$u_x|_{x=0} = u_x|_{x=1} = 0. \quad (3)$$

Here D is a positive-definite diagonal matrix; W is a hysteresis operator which maps an initial configuration function $\xi_0(x)$ ($\in \{1, -1\}$) and an input function $u(x, \cdot)$ to an output function $W(\xi_0(x), u(x, \cdot))(t)$. As a function of (x, t) , $W(\xi_0, u)$ takes values in a set $\mathcal{W} \subset \mathbb{R}^m$ ($m \in \mathbb{N}$). Now we shall define this operator in detail.

Let $\Gamma_\alpha, \Gamma_\beta \subset \mathcal{U}$ be two disjoint smooth manifolds of codimension one without boundary (hysteresis “thresholds”). For simplicity, we assume that they are given by

$\gamma_\alpha(u) = 0$ and $\gamma_\beta(u) = 0$ with $\nabla\gamma_\alpha(u) \neq 0$ and $\nabla\gamma_\beta(u) \neq 0$, respectively, where γ_α and γ_β are C^∞ -smooth functions (in the general situation, atlases can be used).

Denote $M_\alpha = \{u \in \mathcal{U} : \gamma_\alpha(u) \geq 0\}$, $M_\beta = \{u \in \mathcal{U} : \gamma_\beta(u) \leq 0\}$, $M_{\alpha\beta} = \{u \in \mathcal{U} : \gamma_\alpha(u) < 0, \gamma_\beta(u) > 0\}$. Assume that $M_\alpha \cap \Gamma_\beta = \emptyset$ and $M_\beta \cap \Gamma_\alpha = \emptyset$ (Fig. 1).

Next, we introduce locally Hölder continuous functions (*hysteresis “branches”*)

$$W_1 : D(W_1) = M_\alpha \cup \overline{M}_{\alpha\beta} \rightarrow \mathcal{W}, \quad W_{-1} : D(W_{-1}) = M_\beta \cup \overline{M}_{\alpha\beta} \rightarrow \mathcal{W}.$$

We fix $T > 0$ and denote by $C_r[0, T)$ the space of functions which are continuous on the right in $[0, T)$. For any $\zeta_0 \in \{1, -1\}$ (*initial configuration*) and $u_0 \in C([0, T]; \mathcal{U})$ (*input*), we introduce the *configuration function*

$$\zeta : \{1, -1\} \times C([0, T]; \mathcal{U}) \rightarrow C_r[0, T), \quad \zeta(t) = \zeta(\zeta_0, u_0)(t)$$

as follows. Let $X_t = \{s \in (0, t] : u_0(s) \in \Gamma_\alpha \cup \Gamma_\beta\}$. Then $\zeta(0) = 1$ if $u_0(0) \in M_\alpha$, $\zeta(0) = -1$ if $u_0(0) \in M_\beta$, $\zeta(0) = \zeta_0$ if $u_0(0) \in M_{\alpha\beta}$; for $t \in (0, T]$, $\zeta(t) = \zeta(0)$ if $X_t = \emptyset$, $\zeta(t) = 1$ if $X_t \neq \emptyset$ and $u_0(\max X_t) \in \Gamma_\alpha$, $\zeta(t) = -1$ if $X_t \neq \emptyset$ and $u_0(\max X_t) \in \Gamma_\beta$ (Fig. 1).

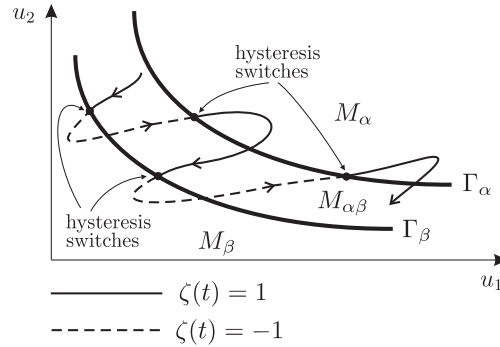


Figure 1: Regions of different behavior of hysteresis W

Now we introduce the *hysteresis operator* $W : \{1, -1\} \times C([0, T]; \mathcal{U}) \rightarrow C_r[0, T)$ by the following rule (cf. [9, 10]). For any initial configuration $\zeta_0 \in \{1, -1\}$ and input $u_0 \in C([0, T]; \mathcal{U})$, the function $W(\zeta_0, u_0) : [0, T] \rightarrow \mathcal{W}$ (*output*) is given by

$$W(\zeta_0, u_0)(t) = W_{\zeta(t)}(u_0(t)), \quad (4)$$

where $\zeta(t)$ is the configuration function defined above.

Now assume that the initial configuration and the input function depend on spatial variable $x \in [0, 1]$. Denote them by $\xi_0(x)$ and $u(x, t)$, where $u(x, \cdot) \in C([0, T]; \mathcal{U})$. Let $u(x, 0) = \varphi(x)$ for $x \in [0, 1]$.

Condition 1. The initial data $\varphi(x)$ and $\xi_0(x)$ are consistent in the following sense. For any $x \in [0, 1]$, we have $\xi_0(x) = 1$ if $\varphi(x) \in M_\alpha$, $\xi_0(x) = -1$ if $\varphi(x) \in M_\beta$, and $\xi_0(x) \in \{1, -1\}$ if $\varphi(x) \in M_{\alpha\beta}$.

Now, using (4), we can define the function

$$W(\xi_0(x), u(x, \cdot))(t), \quad (5)$$

where x is treated as a parameter. We call this function *spatially distributed hysteresis* and the function $\xi(x, t) = \zeta(\xi_0(x), u(x, \cdot))(t)$ its *spatial configuration*.

Definition 1. We say that a function $\varphi \in C^1([0, 1]; \mathcal{U})$ is *transverse* with respect to a spatial configuration $\xi_0(x)$ if it is consistent with $\xi_0(x)$ and the following holds. If $\varphi(\bar{x}) \in \Gamma_\alpha$ (Γ_β) and the vector $\varphi'(\bar{x})$ is tangent to Γ_α (Γ_β) at some point $\bar{x} \in [0, 1]$, then $\xi_0(\bar{x}) = 1$ (-1) in a neighborhood of \bar{x} .

Condition 2. The initial function $\varphi(x)$ is transverse with respect to $\xi_0(x)$.

Remark 1. Suppose that $\varphi(\bar{x}) \in \Gamma_\alpha$. Then the tangency of the vector $\varphi'(\bar{x})$ to the manifold Γ_α can be analytically written as

$$(\nabla \gamma_\alpha|_{u=\varphi(\bar{x})}, \varphi'|_{x=\bar{x}}) = \frac{d}{dx} \gamma_\alpha(\varphi(x)) \Big|_{x=\bar{x}} = 0,$$

where (\cdot, \cdot) is the inner product in \mathbb{R}^k .

In this paper, we shall deal with spatially distributed hysteresis whose spatial configuration has finitely many discontinuity points in x for each t .

Condition 3. There exist points $0 = \bar{b}_0 < \bar{b}_1 < \dots < \bar{b}_M < \bar{b}_{M+1} = 1$ ($M \geq 1$) such that $\xi_0(x) = (-1)^i$ (or $(-1)^{i+1}$) for $x \in (\bar{b}_i, \bar{b}_{i+1})$, $i = 0, \dots, M$.

Now we consider time-dependent functions $u(x, t)$ such that $u, u_x \in C(\overline{Q}_T; \mathcal{U})$.

Definition 2. We say that a function u is *transverse on* $[0, T]$ (with respect to a spatial configuration $\xi(x, t)$) if, for every fixed $t \in [0, T]$, the function $u(\cdot, t)$ is transverse with respect to the spatial configuration $\xi(\cdot, t)$.

Definition 3. We say that a function u *preserves spatial topology* (of a spatial configuration $\xi(x, t)$) on $[0, T]$ if there is $M > 0$ such that, for $t \in [0, T]$, there are continuous functions $0 \equiv b_0(t) < b_1(t) < \dots < b_M(t) < b_{M+1}(t) \equiv 1$ such that $\xi(x, t) = (-1)^i$ (or $(-1)^{i+1}$, respectively) for $x \in (b_i(t), b_{i+1}(t))$, $i = 0, \dots, M$.

The functions $b_i(t)$ play a role of free boundary, which has much in common with free boundary in parabolic obstacle problems (see, e.g., [11, 12] and the references therein). However, in our case, the behaviour of $b_i(t)$ is defined differently.

2. Functional spaces for solutions. We denote by $L_q(0, 1)$, $q > 1$, the standard Lebesgue space and by $W_q^l(0, 1)$ with natural l the standard Sobolev space. For a noninteger $l > 0$, denote by $W_q^l(0, 1)$ the Sobolev space with the norm

$$\|\varphi\|_{W_q^{[l]}} + \left(\int_0^1 dx \int_0^1 \frac{|\varphi^{([l])}(x) - \varphi^{([l])}(y)|^q}{|x - y|^{1+q(l-[l])}} dy \right)^{1/q},$$

where $[l]$ is the integer part of l . Further, we introduce the anisotropic Sobolev spaces $W_q^{2,1}(Q_T)$ with the norm $\left(\int_0^T \|u(\cdot, t)\|_{W_q^2}^q dt + \int_0^T \|u_t(\cdot, t)\|_{L_q}^q dt\right)^{1/q}$ and $W_\infty^{0,1}(Q_T)$ with the norm $\|v\|_{L_\infty(Q_T)} + \|v_t\|_{L_\infty(Q_T)}$. We denote by $C^\lambda(\overline{Q_T})$, $0 < \lambda < 1$, the Hölder space.

For the vector-valued functions, we use the following notation. For example, if $u(x, t) \in \mathcal{U}$ and each component of u belongs to $W_q^{2,1}(Q_T)$, then we write $u \in W_q^{2,1}(Q_T; \mathcal{U})$.

Throughout the paper, we fix q and λ such that $q \in (3, \infty)$ and $\lambda \in (0, 1 - 3/q)$. In particular, this implies that $u, u_x \in C^\lambda(\overline{Q}_T; \mathcal{U})$ whenever $u \in W_q^{2,1}(Q_T; \mathcal{U})$ (see Lemma 3.3 in [13, Chap. 2]).

To define the space of initial data, we use the fact that if $u \in W_q^{2,1}(Q_T; \mathcal{U})$, then the trace $u|_{t=0}$ is well defined and belongs to $W_q^{2-2/q}((0,1); \mathcal{U})$ (see Lemma 2.4 in [13, Chap. 2]). Moreover, one can define the space $W_{q,N}^{2-2/q}((0,1); \mathcal{U})$ as the subspace of functions from $W_q^{2-2/q}((0,1); \mathcal{U})$ with the zero Neumann boundary conditions.

We assume that $\varphi \in W_{q,N}^{2-2/q}((0,1);\mathcal{U})$ and $\psi \in L_\infty((0,1);\mathcal{V})$ in (2).

Definition 4. A pair $(u(x, t), v(x, t))$ is called a *solution of problem (1)–(3) (in Q_T)* if $u \in W_q^{2,1}(Q_T; \mathcal{U})$, $v \in W_\infty^{0,1}(Q_T; \mathcal{V})$; $W(\xi_0, u)$ is measurable with respect to (x, t) ; u and v satisfy equations (1) for a.e. $(x, t) \in Q_T$; the first condition in (2) and conditions (3) are satisfied in the sense of traces; the second condition in (2) holds in the following sense: $v(\cdot, t) \rightarrow \psi(\cdot)$ as $t \rightarrow 0$ in $L_\infty((0, 1); \mathcal{V})$.

The solution is called *transverse (preserving spatial topology)* if $u(x, t)$ is transverse (preserving spatial topology).

It follows from this definition that any solution $u(x, t)$ belongs to $C(\overline{Q}_T; \mathcal{U})$. Therefore, the function $W(\xi_0, u)$ is well defined by (5) and belongs to $L_\infty(Q_T; \mathcal{W})$.

3. Assumptions on the right-hand side. First, we assume the following.

Condition 4. *The functions $f(u, v, w)$ and $g(u, v, w)$ are locally Lipschitz continuous in $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$.*

Next, we formulate dissipativity conditions for f and g .

In the following condition, we denote by \mathcal{U}_μ , $\mu \geq 0$, parallelepipeds in \mathcal{U} with the edges parallel to respective coordinate axes such that $\varphi(x) \in \mathcal{U}_\mu$ for all $x \in [0, 1]$.

Condition 5. *There is a parallelepiped \mathcal{U}_0 and, for each sufficiently small $\mu > 0$, there is a parallelepiped \mathcal{U}_μ and a locally Lipschitz continuous function $f_\mu(u, v)$ such that*

1. $|f_\mu(u, v)|$ converges to 0 uniformly on compact sets in $\mathcal{U} \times \mathcal{V}$ as $\mu \rightarrow 0$,
2. At each point $u \in \partial U_0 \cap D(W_{\pm 1})$, $v \in \mathcal{V}$, the vector $f(u, v, W_{\pm 1}(u)) + f_\mu(u, v)$ points strictly inside \mathcal{U}_0 .
3. At each point $u \in \partial U_\mu \cap D(W_{\pm 1})$, $v \in \mathcal{V}$, the vector $f(u, v, W_{\pm 1}(u_\mu)) + f_\mu(u, v)$ points strictly inside \mathcal{U}_μ for all $u_\mu \in \mathcal{U}_\mu$.

To formulate the assumption on g , we fix some numbers $T_0 > 0$ and $M_0 > 0$. Consider arbitrary functions $\psi(x) \in \mathcal{V}$, $u_0(x, t) \in \mathcal{U}$, $w_0(x, t) \in \mathcal{W}$ such that

$$\|\psi\|_{L_\infty((0,1);\mathcal{V})} \leq M_0, \quad \|u_0\|_{L_\infty(Q_{T_0};\mathcal{U})} \leq M_0, \quad \|w_0\|_{L_\infty(Q_{T_0};\mathcal{W})} \leq M_0.$$

It was proved in [14, Theorem 1, p. 111] that the system of ordinary differential equations (with x being a parameter)

$$v_t = g(u_0(x, t), v, w_0(x, t)), \quad v|_{t=0} = \psi(x) \quad (6)$$

either has a unique solution $v(x, t)$, $\|v\|_{L_\infty(Q_{T_0}; \mathcal{V})} < \infty$, or there is a maximal existence time $T_{max} \in (0, T_0)$ such that $\|v(\cdot, t)\|_{L_\infty((0,1); \mathcal{V})} \rightarrow \infty$ as $t \rightarrow T_{max}$.

Condition 6. *For any $T_0 > 0$ and $M_0 > 0$, there is a compact $\mathcal{V}_0 = \mathcal{V}_0(T_0, M_0) \subset \mathcal{V}$ which does not depend on ψ, u_0, w_0 such that the solution $v(x, t)$ of system (6) satisfies $v(x, t) \in \mathcal{V}_0$ for all $(x, t) \in Q_{T_0}$.*

Remark 2. In particular, the uniform boundedness of v holds if $|g(u, v, w)| \leq A(u, w)|v| + B(u, w)$, where $A(u, w)$ and $B(u, w)$ are bounded on compact sets (see Example 1). However, if $\mathcal{V} \neq \mathbb{R}^l$, one must additionally check that v does not reach the boundary of \mathcal{V} in finite time. To fulfill Condition 6, one could alternatively assume the existence of invariant parallelipipeds for g (similarly to Condition 5).

Note that a solution need not be topology preserving inside its maximal interval of transverse existence. However, if a transverse solution preserves spatial topology and is unique, then it continuously depends on the initial data φ , ψ , and ξ_0 .

Now we discuss the uniqueness of solutions. Let \mathcal{U}_0 be the set from Condition 5.

Condition 7. *There are numbers $K > 0$ and $\sigma \in [0, 1)$ such that*

$$\begin{aligned} |W_1(u) - W_1(\hat{u})| &\leq \frac{K}{(\gamma_\beta(u))^\sigma + (\gamma_\beta(\hat{u}))^\sigma} |u - \hat{u}| & \forall u, \hat{u} \in M_\alpha \cap \overline{M}_{\alpha\beta}, \\ |W_{-1}(u) - W_{-1}(\hat{u})| &\leq \frac{K}{(\gamma_\alpha(u))^\sigma + (\gamma_\alpha(\hat{u}))^\sigma} |u - \hat{u}| & \forall u, \hat{u} \in M_\beta \cap \overline{M}_{\alpha\beta}. \end{aligned}$$

Theorem 3 (uniqueness). *Assume additionally that Condition 7 holds. Let (u, v) and (\hat{u}, \hat{v}) be two transverse solutions of problem (1)–(3) in Q_T for some $T > 0$. Then $(u, v) = (\hat{u}, \hat{v})$.*

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